Effective Critical Exponents from Finite Temperature Renormalization Group

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Abstract

Effective critical exponents for the $\lambda\phi^4$ scalar field theory are calculated as a function of the renormalization group block size k_o^{-1} and inverse critical temperature β_c. Exact renormalization group equations are presented up to first order in the derivative expansion and numerical solutions are obtained with and without polynomial expansion of the blocked potential. For a finite temperature system in d dimensions, it is shown that $\bar{\beta}_c = \beta_c k_o$ determines whether the d-dimensional ($\bar{\beta}_c \ll 1$) or $(d+1)$-dimensional ($\bar{\beta}_c \gg 1$) fixed point governs the phase transition. The validity of a polynomial expansion of the blocked potential near criticality is also addressed.

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The calculation of critical exponents using renormalization group (RG) methods has a long standing history. For the scalar $\lambda\phi^4$ theory in four dimensions it has been shown that only a trivial Gaussian fixed point exists, whereas for $2 \leq d < 4$ a strong-coupling Wilson-Fisher fixed point appears [1]. Studies of the transition from four- to three-dimensional systems typically rely on the ϵ expansion [2]; however, through years of labor it has been shown that this expansion is asymptotic, requiring complicated resummations or outright truncation of the series. In the last few years a new tool has emerged for studying phase transitions in arbitrary dimension: the numerical and/or analytical solution of exact renormalization group equations [3–6]. When solving RG equations one conventionally Taylor expands the blocked potential and solves for the flow of the Taylor coefficients; however, we find evidence that Taylor expansion of the blocked potential near criticality is not justified. In this Letter, we will present results obtained [7] solving finite temperature RG equations numerically without polynomial expansion.

For generality we first consider a finite temperature system defined on the manifold $S^1 \times R^d$ with the radius of S^1 equal to the inverse temperature β. As $T \to 0$, the radius of S^1 goes to infinity making the manifold equivalent to R^{d+1}, while in the opposite limit, $T \to \infty$, the manifold becomes equivalent to R^d. However, as we will show below, the critical behavior of a finite temperature system is actually parameterized by $\bar{\beta}_c = \beta_c k_o$, where k_o is the renormalization blocking scale. The critical behavior of the system will be characteristic of $(d+1)$-dimensions when $\bar{\beta}_c \gg 1$ and d-dimensions when $\bar{\beta}_c \ll 1$, so that in the true thermodynamic limit, $k_o \to 0$, the system will exhibit d-dimensional critical behavior for all critical temperatures $T_c > 0$.

The bare action for a scalar field theory is given by

$$S[\phi] = \int_0^\beta d\tau \int d^d\mathbf{x} \left\{ \frac{Z_\tau}{2} (\partial_\tau \phi)^2 + \frac{Z_s}{2} (\nabla\phi)^2 + V(\phi) \right\}, \tag{1}$$

where we have set $\hbar = 1$, and Z_τ and Z_s are the tree-level temporal and spatial wavefunction renormalization constants, respectively. Note that two wavefunction renormalization constants are required at finite temperature due to the breaking of Lorentz invariance.

The RG evolution of the finite temperature blocked action $\tilde{S}_{\beta,k}[\Phi]$ can be obtained from

$$\mathrm{e}^{-\tilde{S}_{\beta,k}[\Phi]} = \int D[\phi] \prod_{\mathbf{x}} \delta(\phi_{k,\tilde{n}}(\mathbf{x},\tau) - \Phi(\mathbf{x},\tau)) \, \mathrm{e}^{-S[\phi]}, \tag{2}$$

where $\phi_{k,\tilde{n}}(\mathbf{x},\tau)$ is the blocked quantum field, which in general can be expressed in momentum space as

$$\phi_{k,\tilde{n}}(\mathbf{p},\omega_n) = \rho^{(d)}_{k,\tilde{n}}(\mathbf{p},\omega_n) \, \phi(\mathbf{p},\omega_n), \tag{3}$$

with $\rho^{(d)}_{k,\tilde{n}}(\mathbf{p},\omega_n)$ being the blocking function for the system. We choose the blocking function to be

$$\rho^{(d)}_{k,\tilde{n}}(\mathbf{p},\omega_n) = \delta_{n=0} \, \Theta(k - |\,\mathbf{p}|), \tag{4}$$

so that only the modes with $|\mathbf{p}| > k$ and $n = 0$ are included in the blocked field with the other modes being integrated over in the path integral. As for the spatial component of the blocking function, a sharp cutoff cleanly separates the high- and low-momentum modes and

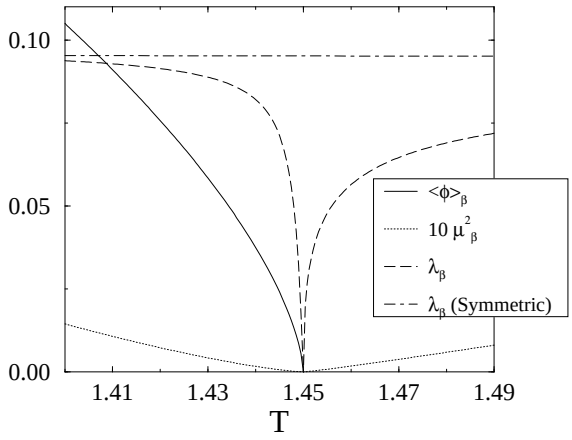

FIG. 1. Temperature dependence of the thermal mass parameter μ_β^2, coupling constant λ_β, and the expectation value $\langle\phi\rangle_\beta$ near $T = T_c$. The T dependence of λ_β in the symmetric phase is also illustrated.

does not introduce any dependence on the shape of the blocking function, as is the case for a smooth cutoff.

In order to derive the coupled RG equations for the blocked potential and wavefunction renormalization we evaluate (2) at one-loop level and expand the background field as $\Phi(\mathbf{x}) = \Phi_o + \tilde{\Phi}(\mathbf{x})$. Derivative terms are handled by treating the fluctuation field, $\tilde{\Phi}(\mathbf{x})$, and the momentum, $\mathbf{p}$, as operators obeying the commutation relation $\left[p_i, \tilde{\Phi}(\mathbf{x})\right] = i\partial_i\tilde{\Phi}(\mathbf{x})$ [8].

We note here that since $\Phi(\mathbf{x})$ is independent of τ we can neglect the RG flow of Z_τ and make the notational identification $Z \equiv Z_s$ throughout the rest of the Letter. In terms of the bare potential $V(\Phi)$ and bare wavefunction renormalization Z the one-loop contributions to the blocked potential and wavefunction renormalization are

$$\begin{aligned}\tilde{U}^{(1)}_{\beta,k} &= \frac{1}{\beta}\int_{|\mathbf{p}|=k}^{\Lambda} \ln\left[\sinh\left(\frac{\beta u_{\mathbf{p},\Phi}}{2}\right)\right] \\ \tilde{\mathcal{Z}}^{(1)}_{\beta,k} &= \frac{Z(V''')^2}{2\beta}\sum_{n=-\infty}^{\infty}\int_{|\mathbf{p}|=k}^{\Lambda}\frac{-Z\mathbf{p}^2/3 + \omega_n + V''}{(\omega_n^2 + u^2_{\mathbf{p},\Phi})^4}\,,\end{aligned} \tag{5}$$

where $u_{\mathbf{p},\Phi} \equiv (Z\,\mathbf{p}^2 + V''(\Phi))^{1/2}$, and Λ is the ultraviolet cutoff and a tilde indicates a perturbative quantity.

Due to their perturbative nature, these equations will fail to capture the true physics near criticality. This is because as the infrared (IR) cutoff, k, is lowered, the loop expansion of the path integral will no longer be expanding about the true minimum of the blocked action but instead about the minimum of the bare action. As an improvement, we can take advantage of the arbitrariness of the IR cutoff and divide the integration into a large number of thin shells. For instance, we could integrate eq. (5) from the UV cutoff, Λ, down to $\Lambda - \Delta k$ and use the potential obtained, $\tilde{U}^{(1)}_{\beta,\Lambda-\Delta k}$, as the initial condition for the integration of the next shell. By dividing the integation into thin shells and taking the limit as the shell width Δk goes to zero, eq. (5) can be made self-consistent allowing a non-perturbative solution of the path integral. This procedure is equivalent to replacing the bare potential, V, and bare wavefunction renormalization, Z, with their k dependent counterparts $U_{\beta,k}$ and $\mathcal{Z}_{\beta,k}$ on the

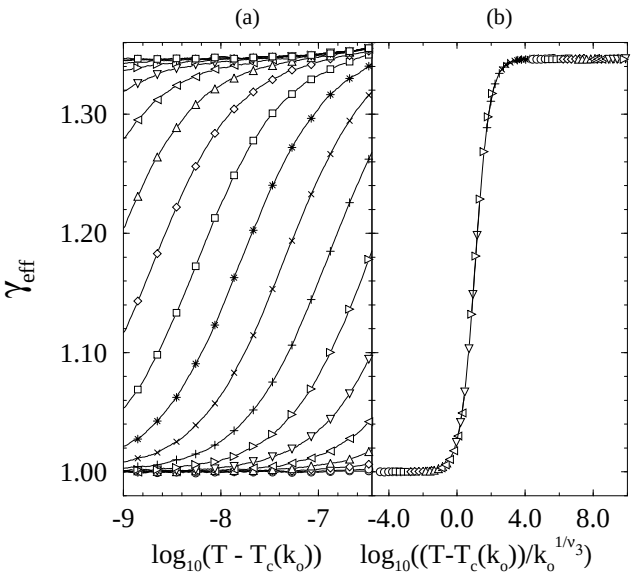

FIG. 2. Effective critical exponent $\gamma_{\rm eff}$ as function of (a) $\log_{10}(T - T_c(k_o))$ and (b) $\log_{10}((T - T_c(k_o))/k_o^{1/\nu_3})$ at Φ^8 level of truncation. The different symbols correspond to different values of k_o.

right hand side of (5), giving

$$U^{(1)}_{\beta,k} = \frac{1}{\beta}\int_{|\mathbf{p}|=k}^{\Lambda} \ln\left[\sinh\left(\frac{\beta u_{\mathbf{p},\Phi}}{2}\right)\right]$$
$$\mathcal{Z}^{(1)}_{\beta,k} = \frac{\mathcal{Z}_{\beta,k}(U'''_{\beta,k})^2}{2\beta}\sum_{n=-\infty}^{\infty}\int_{|\mathbf{p}|=k}^{\Lambda}\frac{-\mathcal{Z}_{\beta,k}\mathbf{p}^2/3 + \omega_n + U''_{\beta,k}}{(\omega_n^2 + u^2_{\mathbf{p},\Phi})^4}\,, \tag{6}$$

with $u_{\mathbf{p},\Phi}$ now given by $u_{\mathbf{p},\Phi} = (\mathcal{Z}_{\beta,k}\,\mathbf{p}^2 + U''_{\beta,k}(\Phi))^{1/2}$. These equations are exact to all loop orders since higher-loop contributions are suppressed by additional powers of Δk, allowing us to sum over all non-overlapping graphs including daisy and super-daisy diagrams [9].

Since eq. (6) contains two scales, k and β, there are two ways to proceed at this stage. The first would be to fix k and differentiate (6) with respect to the temperature; we call this the temperature RG [3]. We will not discuss this method here but instead will concentrate on the other scale, k, which produces the momentum RG. After performing the Matsubara sums [7] in (6) and differentiating the coupled integral equations with respect to the k we have

$$k\frac{dU_{\beta,k}}{dk} = -\frac{S_d k^d}{\beta}\ln\left[\sinh\left(\frac{\beta u_{\mathbf{p},\Phi}}{2}\right)\right]$$
$$k\frac{d\mathcal{Z}_{\beta,k}}{dk} = -\frac{\mathcal{Z}_{\beta,k}(U'''_{\beta,k})^2 S_d k^d}{48 u^7_{k,\Phi}}\left\{(-\mathcal{Z}k^2 + 9U''_{\beta,k})\left[\frac{1}{2}n_{b,k}(1+\beta u_{k,\Phi}(1+n_{b,k}))\right] + \right.$$
$$\left. n_{b,k}(1+n_{b,k})\beta^2 u^2_{k,\Phi}\left[(-\mathcal{Z}k^2 + 3U''_{\beta,k})(1+2n_{b,k}) - \frac{2}{3}\mathcal{Z}k^2\beta u_{k,\Phi}(1+6n_{b,k}+6n_b,k^2)\right]\right\}\,, \tag{7}$$

where $n_{b,k} = (\mathrm{e}^{\beta u_{k,\Phi}} - 1)^{-1}$ and $S_d = 2/(4\pi)^{d/2}\Gamma(d/2)$. With the initial condition $U_{\beta,\Lambda}(\Phi) = V(\Phi)$ these equations allow us to integrate down from $k = \Lambda$ to $k = 0$ for a fixed external temperature keeping track of not only the relevant but the irrelevant coupling constants as well.

$g^{(2m)}_{\beta,k} \sim k^{a_m}$	a_m
$g^{(2)}_{\beta,k}$	$1.998 \pm 0.005 = \gamma_{\rm eff}/\nu_{\rm eff}$
$g^{(4)}_{\beta,k}$	$0.997 \pm 0.004 = \zeta_{\rm eff}/\nu_{\rm eff}$
$g^{(6)}_{\beta,k}$	0.002 ± 0.004
$g^{(8)}_{\beta,k}$	-1.001 ± 0.005
$g^{(10)}_{\beta,k}$	-2.002 ± 0.005
$<\Phi>$	$0.51 \pm 0.01 = \beta_{\rm eff}/\nu_{\rm eff}$

TABLE I. k dependence of vertex functions at the critical point neglecting the effects of wave-function renormalization.

Typically, the first step made in solving equations like (7) is to expand $U_{\beta,k}$ as

$$U_{\beta,k}(\Phi) = \sum_{m=1}^{\infty} \frac{g^{(2m)}_{\beta,k}}{(2m)!}\Phi^{2m}, \quad g^{(2m)}_{\beta,k} = U^{(2m)}_{\beta,k}(0)\,, \tag{8}$$

and solve the coupled ordinary differential equations for the coefficients $g^{(2m)}_{\beta,k}$ after a truncation of the series. However, it is also possible to solve the coupled equations (7) without making any polynomial expansion or truncation. To accomplish this we simply discretize $U_{\beta,k}$ and $\mathcal{Z}_{\beta,k}$ in Φ and k and use finite differences to advance the blocked potential from the bare form to the renormalized form. It is desirable to solve (7) without polynomial truncation since in general the potential will be a non-analytic function of Φ at the critical point [1,4] as well as to provide a consistency check of the polynomial expansion.

In Figure 1 we have plotted the temperature dependence of the renormalized mass, coupling constant, and expectation value, $\hat{\Phi}$, for $d = 3$. The Figure shows that all three of these quantities vanish at T_c continuously indicating a second-order phase transition. Exactly how these quantities vanish determines the critical exponents. Since we have control over both the temperature and k, we can measure the critical exponents in two ways. The first method is to integrate eq. (7) down to $k = k_o$ and measure the temperature dependence of the thermal parameters near $T_c(k_o)$ which is defined by the condition $\mu^2_{\beta_c(k_o),k_o} = \hat{\Phi}_{\beta_c(k_o),k_o} = 0$. Then using the definitions

$$\begin{aligned}
&\chi^{-1} = \mu^2_{\beta,k_o} \sim |T - T_c(k_o)|^{\gamma_{\rm eff}}\,, \quad && \beta \to \beta_c(k_o)\,,\\
&\eta_{\rm eff} = -\frac{\partial \mathcal{Z}_{\beta_c,k_o}(\hat{\Phi}_{\beta_c,k_o})}{\partial \ln k_o}\,, && \beta = \beta_c(k_o)\,,\\
&\hat{\Phi}_{\beta,k_o} \sim |T - T_c(k_o)|^{\beta_{\rm eff}}\,, && \beta \to \beta_c(k_o)\,,\\
&\hat{\Phi}_{\beta_c(k_o),k_o} \sim h^{1/\delta_{\rm eff}}\,, && h \to 0\,,
\end{aligned} \tag{9}$$

we can determine the critical exponents. The exponents measured in this way will be necessarily k_o dependent, with the true d-dimensional critical exponents given in the limit $k_o \to 0$. For $k_o \neq 0$ the critical exponents measured interpolate between the three- and four-dimensional values.

Another way to determine the critical exponents is to fix T at the true critical temperature $T_c = T_c(k_o = 0)$ and use the k_o dependence of the parameters and finite-size scaling

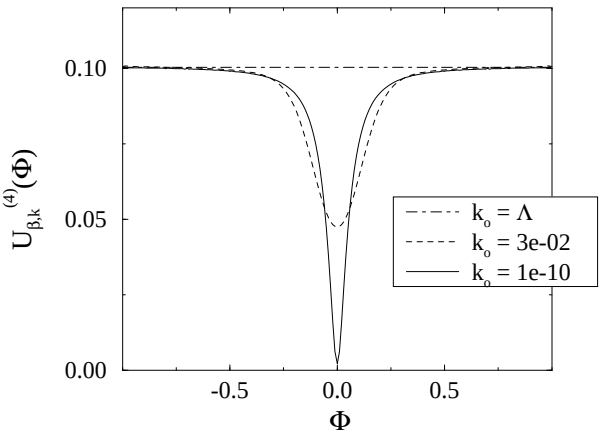

FIG. 3. $U^{(4)}_{\beta,k}$ as a function of Φ close to T_c ($T - T_c(k_o) \sim 10^{-5}$) obtained by solving the RG equations without polynomial truncation.

laws. Finite size scaling states that near T_c the scaling variable is $Y = \ell/\xi$, where ξ is the correlation length of the system and ℓ is the size of the finite dimension. In our analysis there are two finite length scales, the blocking length, k_o^{-1}, and the inverse critical temperature, β_c. The scaling variables associated with these two scales are $Y_\beta = |T - T_c(k_o)|/k_o^{1/\nu_{\rm eff}}$ and $Y_k = (\beta_c k_o)^{1/\nu_{\rm eff}} = \bar{\beta}_c^{1/\nu_{\rm eff}}$, respectively, so that in addition to the standard scaling variable Y_β we get finite size scaling also in the imaginary time direction governed by the dimensionless quantity $\bar{\beta}_c$.

Using finite-size scaling we fix $T = T_c$ in eq. (9) and replace $|T_c - T_c(k_o)|$ with $k_o^{1/\nu_{\rm eff}}$ giving

$$\chi^{-1} = \mu^2_{\beta_c,k_o} \sim k_o^{\gamma_{\rm eff}/\nu_{\rm eff}} , \quad \hat{\Phi}_{\beta_c,k_o} \sim k_o^{\beta_{\rm eff}/\nu_{\rm eff}} . \tag{10}$$

After extracting $\nu_{\rm eff}$ by measuring the k_o dependence of $T_c(k_o)$ ($\lim_{k_o\to 0} |T_c(k_o) - T_c| \sim k_o^{1/\nu_{\rm eff}}$) we can determine the effective critical exponents $\gamma_{\rm eff}$ and $\beta_{\rm eff}$ from finite size data. The results obtained for the finite size scaling exponents are listed in Table I. Together with a measurement of $\nu_{\rm eff}$ these exponents enable us to calculate $\gamma_{\rm eff}$, $\beta_{\rm eff}$, and $\zeta_{\rm eff}$.

In Table II we list the critical exponents we calculated along with previous calculations and experimental values. The values obtained using a polynomial truncation of the blocked potential agree with those obtained in previous works [5,6]; however, the poor agreement with experiment suggests that a polynomial truncation of the blocked potential is not appropriate. This problem has been pointed out by Morris as well [5]. We also observe that the $(2m)$-point functions with $m \geq 3$ diverge with alternating signs $g^{(2m)}_{\beta_c,k} \sim (-1)^{(m+1)} k^{m-3(1+\eta_{\rm eff})}$. This pattern of divergence has been reported elsewhere [4] and provides another indication of a non-polynomial structure for the critical effective potential.

As further evidence for this non-analyticity we can examine the critical effective potential obtained from our non-truncated code. In Fig. 3 we have plotted the fourth derivative of the blocked potential, $U^{(4)}_{\beta,k}$, near T_c for various values of k_o. From this Figure we see that $U^{(4)}_{\beta_c,k\to 0}$ is not identically zero for all Φ as would be predicted by a fourth order polynomial truncation, i.e. $U_{\beta_c,k} = \mu^2_{\beta_c,k}\Phi^2/2 + \lambda_{\beta_c,k}\Phi^4/4!$. Instead, higher order polynomial terms must be introduced in order to describe the shape of $U^{(4)}_{\beta_c,k\to 0}$. For example, a sixth order term

TABLE II. Critical exponents as function of the level of truncation along with the best calculations to date and experimental values. In the four-dimensional limit mean field exponents are obtained at all levels of truncation. NT indicates results obtained without polynomial truncation of the blocked potential. Calculations and experimental values are taken from [1] and [10], respectively.

Measured Critical Exponents for $d_{\rm eff} = 3$					
$O(\Phi^{(2m)})$	γ	ζ (ν)	β	η	δ
2	1.054 ± 0.002	0.527 ± 0.003			
3	1.171 ± 0.004	0.585 ± 0.003			
4	1.345 ± 0.004	0.672 ± 0.004			
5	1.504 ± 0.004	0.744 ± 0.004			
NT	1.234 ± 0.01		0.315 ± 0.007	0.0000	4.65 ± 0.1
NT + $\mathcal{Z}$	1.241 ± 0.008		0.321 ± 0.008	0.036 ± 0.005	4.63 ± 0.1
Best Calculations to Date					
ϵ^5	1.2390 ± 0.0025	0.6310 ± 0.0015	0.3270 ± 0.0025	0.0375 ± 0.0025	4.814 ± 0.015
Experimental Values					
	1.23-1.25	0.624-0.626	0.316-0.327	0.016-0.06	4.6-4.9

would give $U^{(4)}_{\beta_c,k\to 0}$ a parabolic shape near the origin, and higher order terms with increasing magnitude and oscillating signs are needed to fit the plateau at large Φ.

We have demonstrated that it is possible to determine the critical exponents of the scalar $\lambda\phi^4$ theory through numerical integration of RG flow equations without polynomial expansion. This method provides a straightforward alternative to the ϵ expansion and can be used to study field theories in arbitrary dimensions. We have shown through finite-size scaling arguments that the effective critical exponents interpolate smoothly between their $(d+1)-$ and $d-$dimensional values. Our results indicate that the effective potential is a non-analytic function of the background field at the critical point, casting serious doubt on the validity of a polynomial expansion of the blocked potential in the vicinity of the phase transition. In future work we will calculate O(N) exponents and perform studies in two dimensions in order to compare our numerical solutions with exact analytic solutions.

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